

Einstein Metrics on Group Manifolds and Cosets

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ABSTRACT

It is well known that every compact simple group manifold G admits a bi-invariant Einstein metric, invariant under $G_L \times G_R$. Less well known is that every compact simple group manifold except $SO(3)$ and $SU(2)$ admits at least one more homogeneous Einstein metric, invariant still under G_L but with some, or all, of the right-acting symmetry broken. ($SO(3)$ and $SU(2)$ are exceptional in admitting only the one, bi-invariant, Einstein metric.) In this paper, we look for Einstein metrics on three relatively low dimensional examples, namely $G = SU(3)$, $SO(5)$ and G_2 . For $G = SU(3)$, we find just the two already known inequivalent Einstein metrics. For $G = SO(5)$, we find four inequivalent Einstein metrics, thus extending previous results where only two were known. For $G = G_2$ we find six inequivalent Einstein metrics, which extends the list beyond the previously-known two examples. We also study some cosets G/H for the above groups G . In particular, for $SO(5)/U(1)$ we find, depending on the embedding of the $U(1)$, generically two, with exceptionally one or three, Einstein metrics. We also find a pseudo-Riemannian Einstein metric of signature $(2, 6)$ on $SU(3)$, an Einstein metric of signature $(5, 6)$ on $G_2/SU(2)_{\text{diag}}$, and an Einstein metric of signature $(4, 6)$ on $G_2/U(2)$. Interestingly, there are no Lorentzian Einstein metrics among our examples.

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1 Introduction

Finding Einstein metrics on compact spaces is a subject of considerable mathematical interest. It is also of importance in physics, most notably in the compactification of the extra dimensions in supergravity, string theory or M-theory backgrounds. A class of Einstein spaces that was much studied in the 1980’s, prior to the rise of string theory and the consequent emphasis on Ricci-flat Calabi-Yau compactifications, comprised compact homogeneous spaces G/H , admitting a transitive group action under G . Even within the framework of string theory or M-theory, such compactifications still have an important rôle to play, for example in the AdS/CFT correspondence.

Motivated by this, we have looked in a somewhat broader context at the question of the existence of Einstein metrics on compact homogeneous spaces G/H , both for the case where H is some proper subgroup of G and also for the case that H is the identity, in which case the space is just the group manifold G itself. We investigate the first few low-dimensional examples of compact simple groups G . The first non-trivial example, for which the group manifold admits a second Einstein metric, is $SU(3)$. We focus on $SU(3)$, $SO(5)$ and G_2 in our discussions.

It is well known that if g denotes a group element in G then the bi-invariant metric $\text{tr}(g^{-1} dg)^2$ is necessarily Einstein, since all symmetric 2-index tensors that are invariant under $G_L \times G_R$ must be constant multiples of one another. However, this does not exclude the possibility that there could exist further, inequivalent, homogeneous Einstein metrics, invariant still under the action of G_L , but with some or all of the G_R symmetry broken. In fact, it has been shown by D’Atri and Ziller [1] that every compact simple group except $SO(3)$ and $SU(2)$ admits at least one such additional Einstein metric. Such metrics can be constructed as follows. First, we define the left-invariant 1-forms σ_a on G :

$$g^{-1} dg = \sigma_a T^a , \quad (1.1)$$

where T^a are the generators of the Lie algebra of G . Then, the most general left-invariant metric on G can be written as

$$ds^2 = x_{ab} \sigma_a \sigma_b , \quad (1.2)$$

where x_{ab} is a constant symmetric “squashing matrix.” The D’Atri-Ziller examples are obtained by rescaling the bi-invariant metric, for which by a suitable choice for the basis σ_a one may take $x_{ab} = \delta_{ab}$, along a suitably chosen subgroup.

In principle the problem of looking for Einstein metrics within the class (1.2) is a purely mechanical one; first one computes the Ricci tensor R_{ab} as a function of x_{ab} , and then one

solves the algebraic equations resulting from imposing the Einstein condition $R_{ab} = \lambda g_{ab}$.¹ In practice, however, the technical difficulties of solving the equations for the $\frac{1}{2}d(d+1)$ independent components of x_{ab} , where $n = \dim(G)$, can become insurmountable. One option, which is the one we shall follow in this paper, is to make some simplifying assumptions about the structure of x_{ab} , in which many of the components are set to zero, and, possibly, sets of symmetry-related non-zero components are set equal. Thus the general idea is to try various restricted ansätze for the coefficients x_{ab} , motivated by the symmetries of the situation.

By following such a strategy, we succeed in finding four inequivalent Riemannian Einstein metrics on the 10-dimensional group manifold $SO(5)$, of which two appear to be new. We find six inequivalent Riemannian Einstein metrics on the 14-dimensional group manifold G_2 , of which four appear to be new. We also find a pseudo-Riemannian Einstein metric of signature $(2, 6)$ on $SU(3)$, in addition to the two known Riemannian Einstein metrics.

An important question that arises when a candidate “new” Einstein metric is found on a given space is whether it is genuinely inequivalent to previously-obtained metrics. This may not necessarily be easy to see directly, since it might be that some non-trivial change to the basis σ_a would be required in order to reveal the equivalence of two metrics. Although it might, therefore, be quite tricky to demonstrate that two ostensibly different metrics are actually equivalent, the inverse question can often be easily settled. We may consider invariant (*i.e.* dimensionless) quantities, built from the scalar curvature invariants and the magnitude of the volume form of the manifold. If such an invariant takes different values for two metrics, then those metrics are definitely inequivalent. Two such invariants that we find useful, in this regard, are

$$I_1 = \lambda^{d/2} V, \quad I_2 = |\text{Riem}|^2 \lambda^{-2}, \quad (1.3)$$

where λ is the Einstein constant ($R_{ab} = \lambda g_{ab}$), $|\text{Riem}|^2 = R_{abcd} R^{abcd}$, V is the magnitude of the volume form of the manifold, and d is its dimension. By comparing the values of either or both of these invariants for ostensibly different Einstein metrics on a given manifold, one may quickly and unambiguously establish inequivalence, in the event of unequal values, whilst if the invariants take the same values for two metrics this allows one to focus on the these cases for closer examination.

It is also of interest to look for Einstein metrics on the homogeneous spaces G/H . This problem has been studied extensively in the mathematics literature, and also, in dimensions

¹If the metric is to be Riemannian, then x_{ab} should be positive definite. It can sometimes happen that pseudo-Riemannian solutions arise in which x_{ab} is non-singular but indefinite.

such as 7 that are particularly relevant for Kaluza-Klein compactifications, in the physics literature too. We discuss some examples where G is $SU(3)$, $SO(5)$ or G_2 . In particular, we find new Einstein metrics on $SO(5)/U(1)$, and we also find a pseudo-Riemannian Einstein metric of signature $(6, 5)$ on $G_2/SU(2)_{\text{diag}}$, and one of signature $(4, 6)$ on $G_2/U(2)$. Surprisingly, perhaps, we find no Einstein metrics of Lorentzian signature $(1, n)$ on any of the group manifolds or cosets considered in this paper. We shall comment further on this in the conclusion.

2 $SU(3)$ and $SU(3)/SO(3)_{\text{maximal}}$

We identify the Lie algebra of $SU(3)$, denoted by $\mathfrak{su}(3)$, with traceless Hermitean 3×3 matrices $T^A{}_B$, and hence the left-invariant 1-forms $L_A{}^B$ are complex valued, with $L_A{}^A = 0$ and $(L_A{}^B)^\dagger = L_B{}^A$. They satisfy the algebra

$$dL_A{}^B = i L_A{}^C \wedge L_C{}^B. \quad (2.1)$$

It is convenient to decompose the $\mathfrak{su}(3)$ algebra with respect to its maximal $\mathfrak{so}(3)$ subalgebra:

$$\begin{aligned} K_1 &= L_2{}^3 + L_3{}^2, & K_2 &= L_3{}^1 + L_1{}^3, & K_3 &= L_1{}^2 + L_2{}^1, \\ K_4 &= L_1{}^1 - L_2{}^2, & K_5 &= \frac{1}{\sqrt{3}}(L_1{}^1 + L_2{}^2 - 2L_3{}^3), \\ H_1 &= i(L_2{}^3 - L_3{}^2), & H_2 &= i(L_3{}^1 - L_1{}^3), & H_3 &= i(L_1{}^2 - L_2{}^1). \end{aligned} \quad (2.2)$$

The subalgebra $\mathfrak{so}(3)_{\text{maximal}}$ is generated by H_1 , H_2 and H_3 , and the K_i transform as a **5** under $\mathfrak{so}(3)_{\text{maximal}}$.

2.1 Einstein metrics on $SU(3)$

We consider metrics of the form

$$ds_8^2 = x_1(K_1^2 + K_2^2 + K_3^2) + x_2 K_4^2 + x_3 K_5^2 + x_4(H_1^2 + H_2^2 + H_3^2). \quad (2.3)$$

We find two inequivalent Riemannian Einstein metrics, given by

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (1, 1, 1, 1), & \lambda &= \frac{3}{4}, & |\text{Riem}|^2/\lambda^2 &= 8, \\ (x_1, x_2, x_3, x_4) &= (11, 11, 11, 1), & \lambda &= \frac{63}{484}, & |\text{Riem}|^2/\lambda^2 &= \frac{764}{63}. \end{aligned} \quad (2.4)$$

The first is the bi-invariant metric.

We also find a third Einstein metric with the class (2.3), which has indefinite signature (2, 6). This is given (up to scaling) by

$$(x_1, x_2, x_3, x_4) = \left(x_1, -\frac{(1-x_1)(1-5x_1)}{5x_1}, -\frac{(1-x_1)(1-5x_1)}{5x_1}, 1 \right), \quad (2.5)$$

where x_1 is the real root of the cubic equation

$$85x_1^3 - 29x_1^2 + 27x_1 - 3 = 0. \quad (2.6)$$

Since this root is given approximately by $x_1 \approx 0.12130$, it follows from (2.5) that there will be two timelike directions in the metric (2.3) in this case. The Einstein constant is positive, given by

$$\lambda = \frac{3(1-x_1)(10x_1-1)}{20x_1^2(1-5x_1)} \approx 4.848. \quad (2.7)$$

2.2 The five-dimensional coset $SU(3)/SO(3)_{\text{maximal}}$

$SU(3)$ acts on $SU(3)/SO(3)_{\text{maximal}}$, with $SO(3)_{\text{maximal}}$ as a stabiliser. Since the K_i span the tangent space of the coset, any $SU(3)$ -invariant metric must necessarily be invariant under the $SO(3)_{\text{maximal}}$ subgroup. As noted above, the K_i transform as a **5** under $SO(3)_{\text{maximal}}$. This has a unique (up to overall scaling) quadratic invariant, and hence the unique $SU(3)$ -invariant metric on the coset $SU(3)/SO(3)_{\text{maximal}}$ is given by

$$ds_5^2 = K_1^2 + K_2^2 + K_3^2 + K_4^2 + K_5^2. \quad (2.8)$$

Since $SU(3)/SO(3)_{\text{maximal}}$ is a symmetric space, this metric is Einstein, and it is easy to see that

$$\lambda = \frac{3}{2}. \quad (2.9)$$

The metric has no Killing spinors, *i.e.* solutions of $\nabla_a \eta = i/2 \sqrt{\lambda/(d-1)} \Gamma_a \eta$ where Γ_a are the Dirac matrices, obeying the Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2g_{ab}$, and in fact it does not admit a spin structure (see, for example, [2]).

A convenient coordinatisation of the symmetric space $SU(3)/SO(3)$ can be given by defining the coset representative

$$\mathcal{V} = \mathcal{V}_1 \mathcal{V}_2, \quad \mathcal{V}_1 = e^{ix\lambda_1} e^{iy\lambda_4} e^{iz\lambda_6}, \quad \mathcal{V}_2 = e^{i\tilde{\phi}_1 \lambda_3} e^{i\tilde{\phi}_2 \sqrt{3}\lambda_8}, \quad (2.10)$$

where λ_i are the standard Gell-Mann generators for $\mathfrak{su}(3)$, and

$$d\mathcal{V} \mathcal{V}^{-1} = i(P_1 \lambda_1 + P_2 \lambda_4 + P_3 \lambda_6 + P_4 \lambda_3 + P_5 \lambda_8 + Q_1 \lambda_2 + Q_2 \lambda_5 + Q_3 \lambda_7). \quad (2.11)$$

The metric on $SU(3)/SO(3)_{\text{maximal}}$ is then given by

$$ds_5^2 = P_1^2 + P_2^2 + P_3^2 + P_4^2 + P_5^2. \quad (2.12)$$

Defining new azimuthal coordinates

$$\phi_1 = \tilde{\phi}_1 + \tilde{\phi}_2, \quad \phi_2 = \tilde{\phi}_1 - 3\tilde{\phi}_2, \quad (2.13)$$

we find that the basis of 1-forms for the coset is given by

$$\begin{aligned} P_1 &= dx + \frac{1}{2} \sin y \sin 2z \, d\phi_2, \\ P_2 &= \cos x \, dy - \frac{1}{2} \sin x \cos y \sin 2z \, d\phi_2, \\ P_3 &= \cos y \left(\cos x \, dz + \frac{1}{2} \sin x \sin y (3d\phi_1 - \cos 2z \, d\phi_2) \right), \\ P_4 &= \sin 2x \sin y \, dz + \frac{3}{4} \cos^2 y \, d\phi_1 + \frac{1}{8}(3 - \cos 2y) \cos 2z \, d\phi_2, \\ P_5 &= \frac{\sqrt{3}}{8} \left((1 - 3 \cos 2y) \, d\phi_1 + 2 \cos^2 y \cos 2z \, d\phi_2 \right). \end{aligned} \quad (2.14)$$

With the scaling chosen here, the metric is Einstein with $R_{ab} = 6g_{ab}$.

The $SU(3)_{\text{maximal}}$ connection is given by

$$\begin{aligned} Q_1 &= -\cos 2x \sin y \, dz + \frac{3}{4} \sin 2x \cos^2 y \, d\phi_1 + \frac{1}{8}(3 - \cos 2y) \sin 2x \cos 2z \, d\phi_2, \\ Q_2 &= -\sin x \cos y \, dz + \frac{3}{4} \cos x \sin 2y \, d\phi_1 - \frac{1}{4} \cos x \sin 2y \cos 2z \, d\phi_2, \\ Q_3 &= -\sin x \, dy - \frac{1}{2} \cos x \cos y \sin 2z \, d\phi_2. \end{aligned} \quad (2.15)$$

3 Einstein Metrics on the $SO(5)$ Group Manifold

Let L_{AB} be the left-invariant 1-forms of $SO(5)$. They are antisymmetric, $L_{AB} = -L_{BA}$, with $1 \leq A \leq 5$ and $1 \leq B \leq 5$, and they satisfy

$$dL_{AB} = L_{AC} \wedge L_{CB}. \quad (3.1)$$

It is sometimes convenient to define

$$\sigma_i = L_{1i}, \quad \tilde{\sigma}_i = L_{2i}, \quad \nu = L_{12}, \quad \text{where } 3 \leq i \leq 5, \quad 3 \leq j \leq 5. \quad (3.2)$$

We find a total of 4 inequivalent Einstein metrics on $SO(5)$. We obtain these by considering two different classes of metric, associated with two different embeddings of an $SO(3)$ subgroup in $SO(5)$. The first class yields 3 inequivalent Einstein metrics:

3.1 The $SO(3)_{\text{canonical}}$ class

For this class, we make a decomposition in which the subgroup $SO(3)_{\text{canonical}} \subset SO(4) \subset SO(5)$ subgroup, generated by L_{ij} , is manifest:

$$ds_{10}^2 = x_1 \sigma_i^2 + x_2 \tilde{\sigma}_i^2 + x_3 (L_{34}^2 + L_{35}^2 + L_{45}^2) + x_4 \nu^2. \quad (3.3)$$

We may take the magnitude V of the volume form to be defined by $\prod_a e^a = V \prod_{A < B} L_{AB}$, and so

$$V = (x_1 x_2 x_3)^{3/2} x_4^{1/2}. \quad (3.4)$$

We obtain 3 inequivalent Einstein metrics as follows, with the first being the standard bi-invariant metric:

(1) Metric I:

$$(x_1, x_2, x_3, x_4) = (1, 1, 1, 1), \quad \lambda = \frac{3}{2}, \quad I_1 = \frac{243}{32}, \quad I_2 = 10. \quad (3.5)$$

(2) Metric II:

$$(x_1, x_2, x_3, x_4) = \left(\frac{7}{2}, \frac{7}{2}, 1, \frac{19}{4}\right), \quad \lambda = \frac{57}{98}, \quad I_1 = \frac{601692057\sqrt{19}}{421654016}, \quad I_2 = \frac{240}{19}. \quad (3.6)$$

(3) Metric III:

$$(x_1, x_2, x_3, x_4) = (1, 2, 1, 2), \quad \lambda = \frac{9}{8}, \quad I_1 = \frac{59049}{8192}, \quad I_2 = \frac{98}{9}. \quad (3.7)$$

3.2 The $SO(3)_{\text{maximal}}$ class

We can obtain a fourth inequivalent Einstein metric by choosing a basis for the $SO(5)$ left-invariant 1-forms in which the maximal $SO(3)$ subgroup of $SO(5)$ is made manifest. This subgroup is generated by

$$\begin{aligned} Z_8 &= \sqrt{\frac{6}{5}} \left(L_{35} + \frac{1}{\sqrt{3}} (L_{13} + L_{24}) \right), & Z_9 &= \sqrt{\frac{6}{5}} \left(L_{45} + \frac{1}{\sqrt{3}} (L_{23} - L_{14}) \right), \\ Z_{10} &= \sqrt{\frac{2}{5}} (2L_{12} + L_{34}). \end{aligned} \quad (3.8)$$

The remaining generators are

$$\begin{aligned} Z_1 &= \frac{2}{\sqrt{5}} \left(L_{35} - \frac{\sqrt{3}}{2} (L_{13} + L_{24}) \right), & Z_2 &= L_{13} - L_{24}, \\ Z_3 &= \frac{2}{\sqrt{5}} \left(L_{45} - \frac{\sqrt{3}}{2} (L_{23} - L_{14}) \right), & Z_4 &= L_{23} + L_{14}, \\ Z_5 &= \sqrt{\frac{2}{5}} (L_{12} - 2L_{34}), & Z_6 &= \sqrt{2} L_{15}, & Z_7 &= \sqrt{2} L_{25}. \end{aligned} \quad (3.9)$$

In this basis, we consider the class of $SO(5)$ metrics

$$ds_{10}^2 = y_1(Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2) + y_2 Z_5^2 + y_3(Z_6^2 + Z_7^2) + y_4(Z_8^2 + Z_9^2 + Z_{10}^2). \quad (3.10)$$

As well as the “round” Einstein metric $y_1 = y_2 = y_3 = y_4$, which repeats (3.5) above, we obtain a new Einstein metric:

(4) Metric IV:

$$(y_1, y_2, y_3, y_4) = (26, 26, 26, 1), \quad \lambda = \frac{69}{1352}, \quad I_1 = \frac{1564031349}{308915776\sqrt{26}}, \quad I_2 = \frac{16705}{1058}. \quad (3.11)$$

Note that here, we again define V via $\prod_a e^a = V \prod_{A < B} L_{AB}$, and so in this case we have

$$V = 32y_1^2 y_2^{1/2} y_3 y_4^{3/2}. \quad (3.12)$$

Since each of the 4 Einstein metrics (3.5), (3.6), (3.7) and (3.11) has different values for the invariants I_1 and I_2 , they are definitely all inequivalent. Included among them are the standard bi-invariant metric (3.5) and the second Einstein metric (3.11) whose existence was established in [4]. The remaining two Einstein metrics (3.6) and (3.7) appear to be new.

4 Einstein Metrics on Cosets $SO(5)/H$

4.1 Einstein metrics on $SO(5)/U(1)$

Here, we choose the $SO(5)$ basis

$$\begin{aligned} X_1 &= L_{13} + L_{24}, & X_2 &= L_{23} - L_{14}, & X_3 &= L_{13} - L_{24}, & X_4 &= L_{23} + L_{14}, \\ X_5 &= \sqrt{2}L_{15}, & X_6 &= \sqrt{2}L_{25}, & X_7 &= \sqrt{2}L_{35}, & X_8 &= \sqrt{2}L_{45}, \\ X_9 &= cL_{12} + sL_{34}, & X_{10} &= cL_{34} - sL_{12}, \end{aligned} \quad (4.1)$$

where $c = \cos \theta$, $s = \sin \theta$. The two commuting generators are taken to be X_9 and X_{10} . The angle θ parameterises the embedding of the $U(1)$ denominator group in the maximal torus T^2 . The cosets $SO(5)/U(1)$ are obtained by dividing out by X_{10} , and writing the coset metric as

$$ds_9^2 = z_1(X_1^2 + X_2^2) + z_2(X_3^2 + X_4^2) + z_3(X_5^2 + X_6^2) + z_4(X_7^2 + X_8^2) + z_5 X_9^2. \quad (4.2)$$

It appears that the non-trivial range for θ is $0 \leq \theta \leq \pi/4$. Angles outside this range give metrics equivalent to ones with θ inside the range. The actual allowed values of θ for which

the local metrics can be extended smoothly onto complete manifolds will form a discrete but infinite set within the range, characterised by coprime integers (k, ℓ) defining rational numbers k/ℓ .

Solving the Einstein conditions, we find the following:

$\theta = 0 :$ **3 Inequivalent Einstein Metrics:**

$$(z_1, z_2, z_3, z_4, z_5) = (1, 1, \frac{3 - \sqrt{5}}{4}, 1, \frac{3 - \sqrt{5}}{2}), \quad (1, 1, \frac{3 + \sqrt{5}}{4}, 1, \frac{3 + \sqrt{5}}{2}), \\ (1, 1.8522, 1.5490, 0.9614, 3.2786)$$

$\theta = \frac{\pi}{4} :$

1 Inequivalent Einstein Metric

$$(z_1, z_2, z_3, z_4, z_5) = (1, 1.1841, 0.2244, 1.0924, 0.8504)$$

$0 < \theta < \frac{\pi}{4} :$

2 Inequivalent Einstein Metrics; Example for $\theta = \frac{\pi}{6}$:

$$(z_1, z_2, z_3, z_4, z_5) = (1, 1.4222, 1.2123, 0.2888, 1.8813), \quad (1, 1.0950, 0.2049, 1.0476, 0.5375)$$

Note that for $\theta = 0$ we actually obtain 4 solutions for the coefficients, namely the three listed plus a fourth (numerical) solution. This is equivalent, up to permutation of generators, to the third listed solution. For $\theta = \pi/4$, we actually obtain two (numerical) solutions, but the second is equivalent to the one listed. For generic θ , *i.e.* in the range $0 < \theta < \pi/4$, we obtain exactly two (numerical) solutions, and they are inequivalent. According to [3], only one Einstein metric was known previously for each θ .

4.2 Einstein Metrics on $SO(5)/T^2$

To construct these, we begin with the $SO(5)$ basis X_a defined in equation (4.1), and then omit the generators associated with the Cartan subalgebra X_9 and X_{10} . Thus we consider

$$ds_8^2 = w_1 (X_1^2 + X_2^2) + w_2 (X_3^2 + X_4^2) + w_3 (X_5^2 + X_6^2) + w_4 (X_7^2 + X_8^2). \quad (4.3)$$

We obtain two solutions with

$$(w_1, w_2, w_3, w_4) = (\frac{24-4\sqrt{6}}{15}, \frac{24-4\sqrt{6}}{15}, \frac{7-2\sqrt{6}}{5}, 1), \quad (\frac{24+4\sqrt{6}}{15}, \frac{24+4\sqrt{6}}{15}, \frac{7+2\sqrt{6}}{5}, 1), \quad (4.4)$$

and four solutions with

$$(w_1, w_2, w_3, w_4) = (4, 2, 3, 1), \quad (\frac{2}{3}, \frac{4}{3}, \frac{1}{3}, 1), \quad (2, 4, 3, 1), \quad (\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, 1). \quad (4.5)$$

In fact the two solutions in (4.4) are equivalent up to permutation and scaling. Similarly, the four solutions in (4.5) are equivalent up to permutation and scaling. This might be suspected from the values of the invariants

$$I_1 = \lambda^4 V, \quad I_2 = |\text{Riem}|^2 \lambda^{-2}, \quad (4.6)$$

which are given by $(I_1, I_2) = (40/27, 1449/100)$ for both of the solutions in (4.4), and by $(I_1, I_2) = (3/2, 16)$ for all four of the solutions in (4.5). Explicit calculations show that indeed the pair (4.4) can be related by relabelling and scaling, as can the quartet (4.5).

Thus we have in total two inequivalent Einstein metrics on $SO(5)/T^2$. One Einstein metric corresponds to taking either of the equivalent pair in (4.4). The other corresponds to taking any one of the equivalent quadruplet in (4.5).²

4.3 Seven-Dimensional Einstein Spaces $SO(5)/SO(3)$

4.3.1 Einstein metrics on $SO(5)/SO(3)_{\text{canonical}}$

Here, we take the $SO(3)$ subgroup to be generated by the subset L_{ij} , where $3 \leq i \leq 5$ and $3 \leq j \leq 5$. The metric on the coset $SO(5)/SO(3)_{\text{canonical}}$, which is the Stiefel manifold $V_{5,2}$, is then taken to be of the form

$$ds_7^2 = u_1 \sigma_i^2 + u_2 \tilde{\sigma}_i^2 + u_3 \nu^2, \quad (4.7)$$

where as before, $\nu = L_{12}$, $\sigma_i = L_{1i}$ and $\tilde{\sigma}_i = L_{2i}$, where $3 \leq i \leq 5$. We can obtain one Einstein metric in this class, by taking

$$(u_1, u_2, u_3) = (1, 1, \frac{3}{2}). \quad (4.8)$$

It satisfies the Einstein equations $R_{ab} = \lambda g_{ab}$ with

$$\lambda = \frac{9}{4}. \quad (4.9)$$

This metric admits two Killing spinors [6], satisfying

$$\nabla_a \eta - \frac{i}{2} m \Gamma_a \eta = 0, \quad (4.10)$$

where $6m^2 = \lambda = 9/4$.

²Reference [3] states that three inequivalent Einstein metrics on $SO(5)/T^2$ are known. The result is attributed to Sakane [5]. In fact Sakane obtained the two solutions (4.4) and the four solutions (4.5), but did not explicitly discuss equivalences among them. We suspect that the two equivalent solutions (4.4) were mistakenly counted as being distinct.

4.3.2 Einstein metrics on $SO(5)/SO(3)_{\text{maximal}}$

In this case, we take the $SO(3)$ subgroup to be maximal in $SO(5)$. Under this embedding, we have the group decompositions

$$\mathbf{4} \longrightarrow \mathbf{4}, \quad \mathbf{5} \longrightarrow \mathbf{5}, \quad \mathbf{10} \longrightarrow \mathbf{7+3}. \quad (4.11)$$

The subgroup $SO(3)_{\text{maximal}}$ is generated by (Z_8, Z_9, Z_{10}) defined in (3.8). The remaining coset generators are Z_a for $1 \leq a \leq 7$, as defined in (3.9).

There is a unique metric (up to scaling) on coset $SO(5)/SO(3)_{\text{maximal}}$, given by

$$ds_7^2 = Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 + Z_6^2 + Z_7^2. \quad (4.12)$$

It is Einstein, satisfying $R_{ab} = \lambda g_{ab}$ with

$$\lambda = \frac{27}{20}. \quad (4.13)$$

This metric admits one Killing spinor [6].

4.3.3 Einstein metrics on $SO(5)/SO(3)_L$

Here, we take the subgroup $SO(3)_L$ in the isomorphism $SO(4) = SO(3)_L \times SO(3)_R$ as the denominator in the coset. The coset has the topology S^7 . Taking $1 \leq a \leq 4$, we split the $SO(5)$ generators as L_{a5} and L_{ab} , and the decompose L_{ab} into their self-dual and anti-self-dual parts:

$$\begin{aligned} L_1 &= \frac{1}{\sqrt{2}} (L_{12} - L_{34}), & L_2 &= \frac{1}{\sqrt{2}} (L_{23} - L_{14}), & L_3 &= \frac{1}{\sqrt{2}} (L_{31} - L_{24}), \\ R_1 &= \frac{1}{\sqrt{2}} (L_{12} + L_{34}), & R_2 &= \frac{1}{\sqrt{2}} (L_{23} + L_{14}), & R_3 &= \frac{1}{\sqrt{2}} (L_{31} + L_{24}). \end{aligned} \quad (4.14)$$

We then consider metrics

$$ds_7^2 = v_1 (L_{15}^2 + L_{25}^2 + L_{35}^2 + L_{45}^2) + v_2 (R_1^2 + R_2^2 + R_3^2). \quad (4.15)$$

We obtain two inequivalent Einstein metrics, with

$$(v_1, v_2) = (1, 2), \quad (1, \frac{2}{5}). \quad (4.16)$$

These correspond to the round S^7 , and the squashed S^7 of Jensen [7], respectively. They satisfy $R_{ab} = \lambda g_{ab}$ with $\lambda = 3/2$ and $\lambda = 27/10$ respectively.

The round S^7 admits 8 Killing spinors, while the squashed Einstein metric admits 1 Killing spinor [8].

5 Einstein Metrics on G_2

The exceptional group G_2 is a subgroup of $SO(7)$. Let the generators for $SO(7)$ be $T_{AB} = -T_{BA}$. If we decompose the $SO(7)$ fundamental index A as $A = (i, \hat{i}, 7)$, where $i = 1, 2, 3$, $\hat{i} = \hat{1}, \hat{2}, \hat{3} = 4, 5, 6$, then the $14 = 3+3+8$ generators of G_2 can be taken to be [8]

$$\begin{aligned} G_i &= T_{i7} + \frac{1}{2}\epsilon_{ijk}T_{\hat{j}\hat{k}}, & G_{ij} &= T_{ij} + T_{\hat{i}\hat{j}}, \\ G_{i\hat{j}} &= \frac{2}{\sqrt{3}}\left(-T_{i\hat{j}} - \frac{1}{2}T_{j\hat{i}} + \frac{1}{2}\delta_{ij}T_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}T_{\hat{k}7}\right). \end{aligned} \quad (5.1)$$

Note that $G_{i\hat{j}}$ is traceless; $G_{\hat{i}\hat{i}} = 0$.

If we associate left-invariant 1-forms $\sigma_a = \{\sigma_i, \sigma_{ij}, \sigma_{i\hat{j}}\}$ with each G_2 generator, then we may write

$$\begin{aligned} \sigma_i &\cong L_{i7} + \frac{1}{2}\epsilon_{ijk}L_{\hat{j}\hat{k}}, & \sigma_{ij} &\cong L_{ij} + L_{\hat{i}\hat{j}}, \\ \sigma_{i\hat{j}} &\cong \frac{2}{\sqrt{3}}\left(-L_{i\hat{j}} - \frac{1}{2}L_{j\hat{i}} + \frac{1}{2}\delta_{ij}L_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}L_{\hat{k}7}\right), \end{aligned} \quad (5.2)$$

where $L_{AB} = -L_{BA}$ are left-invariant 1-forms for $SO(7)$, satisfying $dL_{AB} = L_{AC}\wedge L_{CB}$. When evaluating the exterior derivatives of σ_a , we then project into the subspace of 2-forms spanned by wedge products of the σ_a , in order to read off the Cartan-Maurer equations for the left-invariant 1-forms of G_2 .³

We can consider the class of left-invariant metrics of the form

$$ds_{14}^2 = \sum_{a=1}^6 x_a E_a^+ E_a^- + x_7 H_1^2 + x_8 H_2^2, \quad (5.3)$$

where E_a^+ denotes the six left-invariant 1-forms corresponding to the six positive roots of G_2 , and correspondingly, E_a^- denotes the six left-invariant conjugate 1-forms for the negative roots. H_1 and H_2 denote the left-invariant 1-forms for the two Cartan generators.

In terms of the left-invariant 1-forms σ_i , σ_{ij} and $\sigma_{i\hat{j}}$ defined above, we have

$$\begin{aligned} E_1^+ &= \sigma_{3\hat{2}} - \sigma_{2\hat{3}} + \frac{i}{\sqrt{3}}(\sigma_{23} - 2\sigma_1), \\ E_2^+ &= \sigma_{3\hat{1}} - \sigma_{1\hat{3}} + \frac{i}{\sqrt{3}}(\sigma_{31} - 2\sigma_2), \\ E_3^+ &= \sigma_{2\hat{1}} - \sigma_{1\hat{2}} - \frac{i}{\sqrt{3}}(\sigma_{12} - 2\sigma_3), \\ E_4^+ &= \sigma_{1\hat{2}} + \sigma_{2\hat{1}} - i\sqrt{3}\sigma_{12}, \\ E_5^+ &= \sigma_{1\hat{3}} + \sigma_{3\hat{1}} + i\sqrt{3}\sigma_{31}, \\ E_6^+ &= \sigma_{2\hat{3}} + \sigma_{3\hat{2}} + i\sqrt{3}\sigma_{23}, \end{aligned} \quad (5.4)$$

³This projection procedure is the implementation, at the level of the exterior algebra of the 1-forms, of the fact that the commutators of the G_2 generators G_i , G_{ij} and $G_{i\hat{j}}$ defined by (5.1) close on themselves.

and

$$H_1 = \sigma_{1\hat{1}} - \sigma_{2\hat{2}}, \quad H_2 = -\sqrt{3}(\sigma_{1\hat{1}} + \sigma_{2\hat{2}}). \quad (5.5)$$

The left-invariant 1-forms E_a^- corresponding to the negative roots are obtained from the E_a^+ in (5.4) by reversing the sign of i . The weights of the E_a^+ under (H_1, H_2) are

$$(1, 1), \quad (1, -1), \quad (0, 2), \quad (2, 0), \quad (1, 3), \quad (1, -3), \quad (5.6)$$

for $a = 1$ up to $a = 6$ respectively.

5.1 The $SU(2)_{\text{diag}}$ class

The metric (5.3) can thus be written as

$$\begin{aligned} ds_{14}^2 = & x_1 \left[(\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2 + \frac{1}{3}(\sigma_{23} - 2\sigma_1)^2 \right] + x_2 \left[(\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 + \frac{1}{3}(\sigma_{31} - 2\sigma_2)^2 \right] \\ & + x_3 \left[(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + \frac{1}{3}(\sigma_{12} - 2\sigma_3)^2 \right] \\ & + x_4 \left[3\sigma_{23}^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2 \right] + x_5 \left[3\sigma_{31}^2 + (\sigma_{3\hat{1}} + \sigma_{1\hat{3}})^2 \right] + x_6 \left[3\sigma_{12}^2 + (\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 \right] \\ & + x_7 (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + 3x_8 (\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2. \end{aligned} \quad (5.7)$$

As will be seen in section 6.2 below, the basis used here is naturally adapted to the embedding of the $SU(2)_{\text{diag}}$ subgroup in G_2 , where $SU(2)_{\text{diag}}$ is the diagonal $SU(2)$ in the $SU(2) \times SU(2)$ subgroup of G_2 .

We find two choices (up to overall scaling) for the coefficients x_i that yield Einstein metrics, namely

$$\begin{aligned} (x_1, \dots, x_8) = (3, 3, 3, 1, 1, 1, 1, 1) : \quad \lambda = \frac{1}{3}, \quad I_1 = \frac{1}{81}, \quad I_2 = 14, \\ (x_1, \dots, x_8) = \left(\frac{11}{3}, \frac{11}{3}, \frac{11}{3}, 1, 1, 1, 1, 1\right) : \quad \lambda = \frac{37}{121}, \quad I_1 = \frac{(37)^7}{27 \cdot (11)^{11}}, \quad I_2 = \frac{19346}{1369}. \end{aligned} \quad (5.8)$$

Here we take the volume to be $V = x_1 x_2 x_3 x_4 x_5 x_6 \sqrt{x_7 x_8}$ when calculating the invariant I_1 . Note that the first metric in (5.8) is the bi-invariant one. The second is the non-bi-invariant metric obtained in the analysis of D'Atri and Ziller [1].

5.2 The $SU(2) \times SU(2)$ class

We can obtain further Einstein metrics on G_2 by considering a different choice of basis for the metric, adapted this time to the $SU(2) \times SU(2)$ subgroup of G_2 (see sections 6.3 and

6.4 below). If we take

$$\begin{aligned}
ds_{14}^2 = & y_1 [(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} - \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2] \\
& + y_2 [(\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2] \\
& + y_3 [(\sigma_{23} - \sigma_1)^2 + (\sigma_{31} - \sigma_2)^2 + (\sigma_{12} - \sigma_3)^2] \\
& + y_4 [(\sigma_{23} + \sigma_1)^2 + (\sigma_{31} + \sigma_2)^2 + (\sigma_{12} + \sigma_3)^2] \\
& + y_5 (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + y_6 (\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2,
\end{aligned} \tag{5.9}$$

then the metric is Einstein if

$$y_1 = 3, \quad y_2 = 1, \quad y_4 = \frac{(7y_3 - 6)(6 - y_3)}{15y_3}, \quad y_5 = 1, \quad y_6 = 3, \tag{5.10}$$

and y_3 is a root of the quartic polynomial

$$(y_3 - 3)(35y_3^3 - 303y_3^2 + 666y_3 - 378) = 0. \tag{5.11}$$

The root $y_3 = 3$ reproduces the first Einstein metric listed in (5.8). The three roots of cubic polynomial factor in (5.11) are given by

$$y_3 = \frac{101}{35} + \frac{2\sqrt{2431}}{35} \cos\left(\frac{\theta + 2\pi n}{3}\right), \quad \text{for } n = 0, 1, 2, \tag{5.12}$$

where

$$\cos \theta = \frac{84671}{(2431)^{3/2}}. \tag{5.13}$$

These roots are all real and positive, and furthermore y_4 , given in (5.10), is positive in all these cases. This yields three further Einstein metrics on the G_2 group manifold, which are all inequivalent, and they are all inequivalent to the two already listed in (5.8). The Einstein constant and the invariants $I_1 = \lambda^7 V$ and $I_2 = \lambda^{-2} |\text{Riem}|^2$ for the three additional Einstein metrics, for $n = 0$, $n = 1$ and $n = 2$ in (5.12), are given numerically by

$$\begin{aligned}
n = 0 : \quad & \lambda \approx 0.40067, \quad I_1 \approx 0.021017, \quad I_2 \approx 20.84408, \\
n = 1 : \quad & \lambda \approx 0.60962, \quad I_1 \approx 0.012100, \quad I_2 \approx 19.35457, \\
n = 2 : \quad & \lambda \approx 0.35162, \quad I_1 \approx 0.036879, \quad I_2 \approx 14.30375
\end{aligned} \tag{5.14}$$

For comparison, the numerical values of the invariants for the two Einstein metrics listed in (5.8) are $(I_1, I_2) \approx (0.037037, 14)$ and $(I_1, I_2) \approx (0.036970, 14.13148)$ respectively.

5.3 The $SU(2)_{\max}$ class

There is a third choice of basis that enables us to find one further inequivalent Einstein metric on G_2 . This basis is adapted to the maximal $SU(2)$ subgroup in G_2 (see section 6.5 below), and in it the metric is given by

$$\begin{aligned} ds_{14}^2 = & x_1 \left[\frac{1}{3}(\sigma_{23} - 2\sigma_1)^2 + (\sigma_{3\hat{2}} - \sigma_{2\hat{3}})^2 \right] + x_2 \left[\frac{1}{3}(\sigma_{31} - 2\sigma_2)^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 \right], \\ & + x_3 \left[3\sigma_{1\hat{2}}^2 + (\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 \right] + x_4 \left[3\sigma_{3\hat{1}}^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 \right], \\ & + x_5 \left[\frac{1}{3} \left(\frac{3\sqrt{3}}{\sqrt{5}} \sigma_{23} - 2\sigma_3 + \sigma_{12} \right)^2 + \left(\frac{\sqrt{3}}{\sqrt{5}} (\sigma_{2\hat{3}} + \sigma_{3\hat{2}}) + \sigma_{1\hat{2}} - \sigma_{2\hat{1}} \right)^2 \right] + \frac{4}{3}x_6 (2\sigma_{1\hat{1}} + 3\sigma_{2\hat{2}})^2 \\ & + \frac{1}{96}x_7 (\sqrt{15} \sigma_{23} + 6\sigma_3 + \sigma_{12})^2 + \frac{1}{288}x_7 (\sqrt{15}(\sigma_{2\hat{3}} + \sigma_{3\hat{2}}) - 9(\sigma_{1\hat{2}} + \sigma_{2\hat{1}}))^2 + \frac{1}{588}x_7 (4\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2. \end{aligned} \quad (5.15)$$

We find two choices of coefficients (up to scale) that give Einstein metrics, namely

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, \frac{1}{3}, \frac{1}{3}, \frac{5}{14}, \frac{3}{28}, 28), \quad (5.16)$$

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (1, 1, \frac{1}{3}, \frac{1}{3}, \frac{5}{14}, \frac{3}{28}, \frac{28}{85}). \quad (5.17)$$

The first case is just the standard bi-invariant metric but (5.17) is new, with

$$\lambda = \frac{26}{17}, \quad I_1 = \frac{(26)^7}{3^4 17^8 5\sqrt{85}}, \quad I_2 = \frac{5719}{260}. \quad (5.18)$$

The invariants I_1 and I_2 are different from those we found in the previous five Einstein metrics.

To summarise, we have found six inequivalent Einstein metrics on the G_2 group manifold, of which the last four, given by (5.12) and (5.17), appear to be new.

6 Einstein Metrics on Cosets G_2/H

6.1 $G_2/(SU(2) \times SU(2))$

The $SU(2) \times SU(2)$ subgroup is generated by

$$X_i = G_i - \frac{1}{2}\epsilon_{ijk}G_{jk}, \quad Y_i = G_i + \frac{1}{2}\epsilon_{ijk}G_{jk}, \quad (6.1)$$

where X_i are the generators of one $SU(2)$ factor, and Y_i generates the other. The space is isotropy-irreducible, and so there is just one Einstein metric [9]. It is given by

$$\begin{aligned} ds_8^2 = & 3[(\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 + (\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2] \\ & + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2 + (\sigma_{3\hat{1}} + \sigma_{1\hat{3}})^2 + (\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 \\ & + 3(\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2 + (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2. \end{aligned} \quad (6.2)$$

It satisfies $R_{ab} = \frac{2}{3}g_{ab}$.

6.2 $G_2/SU(2)_{\text{diag}}$

We can obtain an Einstein metric on the 11-dimensional coset space $G_2/SU(2)_{\text{diag}}$, where $SU(2)_{\text{diag}}$ is the diagonal $SU(2)$ subgroup in $SU(2) \times SU(2)$. It is therefore associated with σ_{12} , σ_{23} and σ_{31} . We can then obtain G_2 invariant metrics on the coset, with

$$\begin{aligned} ds_{11}^2 = & y_1 [(\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 + (\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2] \\ & + \frac{1}{3}y_2 [(\sigma_{23} - 2\sigma_1)^2 + (\sigma_{31} - 2\sigma_2)^2 + (\sigma_{12} - 2\sigma_3)^2] \\ & + y_3 [(\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2 + (\sigma_{3\hat{1}} + \sigma_{1\hat{3}})^2 + (\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + 3(\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2]. \end{aligned} \quad (6.3)$$

We then find that there is a Riemannian Einstein metric if

$$y_2 = \frac{y_1(27 - 5y_1)}{9 + 5y_1}, \quad y_3 = 1, \quad (6.4)$$

where y_1 is the real, positive root of the quartic polynomial

$$125y_1^4 - 500y_1^3 + 213y_1^2 + 378y_1 - 972 = 0. \quad (6.5)$$

There is also a pseudo-Riemannian Einstein metric when y_1 is the real, negative root of (6.5). Since y_2 , given by (6.4), is then also negative, the metric signature is (6, 5).

Since the adjoint of G_2 decomposes under the $SU(2) \times SU(2)$ maximal subgroup as

$$\mathbf{14} \longrightarrow (\frac{3}{2}, \frac{1}{2}) \oplus (1, 0) \oplus (0, 1), \quad (6.6)$$

where we denote an $SU(2)$ representation by its spin j , it follows that under $SU(2)_{\text{diag}}$ we shall have

$$\mathbf{14} \longrightarrow (2) \oplus 3 \times (1). \quad (6.7)$$

(In other words, we have one spin-2 and three spin-1 representations in the decomposition.) One may define the Dynkin index of an $SU(2)$ embedding in a group G by

$$I_D = \frac{1}{8} \sum_j \rho_j, \quad \rho_j = \frac{2}{3}j(j+1)(2j+1), \quad (6.8)$$

where the summation is taken over all the irreducible representations, labelled by their spin j , in the decomposition of the adjoint of G . Thus we see that the Dynkin index for the $SU(2)_{\text{diag}}$ subgroup in G_2 is given by

$$I_D = 4. \quad (6.9)$$

The Riemannian Einstein metric we have obtained here is therefore the one listed as $G_2/SO(3)_4$ in [3], which was obtained in [10].

6.3 $G_2/SU(2)_L$

Here, we consider the coset formed by dividing out by the $SU(2)_L$ factor in the $SU(2)_L \times SU(2)_R$ subgroup described in section 6.1. This amounts to factoring out the three terms proportional to y_4 in (5.9), which can be done provided the relations

$$y_2 = \frac{1}{3}y_1, \quad y_5 = \frac{1}{3}y_1, \quad y_6 = y_1 \quad (6.10)$$

are imposed. Metrics on the eleven-dimensional coset are therefore given by

$$\begin{aligned} ds_{11}^2 = & y_1 [(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} - \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2] \\ & + \frac{1}{3}y_1 [(\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2] \\ & + y_3 [(\sigma_{23} - \sigma_1)^2 + (\sigma_{31} - \sigma_2)^2 + (\sigma_{12} - \sigma_3)^2] \\ & + \frac{1}{3}y_1 (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + y_1 (\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2, \end{aligned} \quad (6.11)$$

Imposing the Einstein condition $R_{ij} = \lambda g_{ij}$, we obtain two solutions (up to overall scale):

$$\begin{aligned} (y_1, y_3) = (1, 2) : \quad \lambda = \frac{5}{4}, \quad I_1 = \frac{3125\sqrt{5}}{512\sqrt{2}}, \quad I_2 = \frac{257}{15}, \\ (y_1, y_3) = (1, \frac{2}{7}) : \quad \lambda = \frac{53}{28}, \quad I_1 = \frac{(53)^{11/2}}{2^{19/2} 7^7}, \quad I_2 = \frac{132517}{8427}. \end{aligned} \quad (6.12)$$

From (6.6) we see that under $SU(2)_L$, the adjoint of G_2 decomposes as

$$\mathbf{14} \longrightarrow 2 \times (\frac{3}{2}) \oplus (1) \oplus 3 \times (0), \quad (6.13)$$

and hence from (6.8) the Dynkin index of the $SU(2)_L$ embedding is

$$I_D = 3. \quad (6.14)$$

The two Einstein metrics we have obtained here are the ones denoted by $G_2/SU(2)_3$ in [3], which were obtained in [7, 11].

6.4 $G_2/SU(2)_R$

Here, we consider the coset formed by dividing out by the $SU(2)_R$ factor in the $SU(2)_L \times SU(2)_R$ subgroup described in section 6.1. This amounts to factoring out the three terms proportional to y_3 in (5.9), which can be done provided the relations

$$y_2 = \frac{1}{3}y_1, \quad y_5 = \frac{1}{3}y_1, \quad y_6 = y_1 \quad (6.15)$$

are imposed. Metrics on the eleven-dimensional coset are therefore given by

$$\begin{aligned} ds_{11}^2 = & y_1 [(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} - \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2] \\ & + \frac{1}{3}y_1 [(\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2] \\ & + y_4 [(\sigma_{23} + \sigma_1)^2 + (\sigma_{31} + \sigma_2)^2 + (\sigma_{12} + \sigma_3)^2] \\ & + \frac{1}{3}y_1 (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + y_1 (\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2, \end{aligned} \quad (6.16)$$

Imposing the Einstein condition $R_{ij} = \lambda g_{ij}$, we obtain two solutions (up to overall scale) with $y_1 = 0$ and $315y_4^2 - 144y_4 + 4 = 0$:

$$\begin{aligned} (y_1, y_4) = (1, \frac{2(12+\sqrt{109})}{105}) : \quad \lambda = \frac{44-\sqrt{109}}{28}, \quad I_1 = \frac{(91-8\sqrt{109})^{11/2}}{1536\sqrt{6}(12-\sqrt{109})}, \\ I_2 = \frac{14448791-425072\sqrt{109}}{2818800}, \\ (y_1, y_4) = (1, \frac{2(12-\sqrt{109})}{105}) : \quad \lambda = \frac{44+\sqrt{109}}{28}, \quad I_1 = \frac{(91+8\sqrt{109})^{11/2}}{1536\sqrt{6}(12+\sqrt{109})}, \\ I_2 = \frac{14448791+425072\sqrt{109}}{2818800}. \end{aligned} \quad (6.17)$$

From (6.6) we see that under $SU(2)_R$, the adjoint of G_2 decomposes as

$$\mathbf{14} \longrightarrow (1) \oplus 4 \times (\frac{1}{2}) \oplus 3 \times (0), \quad (6.18)$$

and hence from (6.8) the Dynkin index of the $SU(2)_R$ embedding is

$$I_D = 1. \quad (6.19)$$

The two Einstein metrics we have obtained here are the ones denoted by $G_2/SU(2)_1$ in [3], which were obtained in [7, 11].

6.5 $G_2/SU(2)_{\max}$

There is one further inequivalent 11-dimensional coset G_2/H that we may consider, for which H is the maximal $SU(2)$ subgroup in G_2 . Under this subgroup, the adjoint decomposes as $\mathbf{14} \longrightarrow \mathbf{11} + \mathbf{3}$, which, in terms of the labelling of $SU(2)$ representations by their spin j , reads

$$\mathbf{14} \longrightarrow (5) \oplus (1). \quad (6.20)$$

From (6.8), it follows that the Dynkin index of this embedding is

$$I_D = 28. \quad (6.21)$$

We find that the (canonically-normalised) left-invariant 1-forms of the $SU(2)_{\max}$ subgroup are defined by the Cartan 1-form H_{\max} and positive-root 1-form E_{\max}^+ , whose expressions in terms of (5.4) and (5.5) are

$$H_{\max} = \frac{1}{28} (5H_1 + H_2), \quad E_{\max}^+ = \frac{1}{14\sqrt{2}} (3E_3^+ + \frac{\sqrt{5}}{\sqrt{3}} E_6^+). \quad (6.22)$$

We accordingly find that one can make a projection into the 11-dimensional coset metric

$$\begin{aligned} ds_{11}^2 = & x_1 \left[\frac{1}{3}(\sigma_{23} - 2\sigma_1)^2 + (\sigma_{3\hat{2}} - \sigma_{2\hat{3}})^2 \right] + x_2 \left[\frac{1}{3}(\sigma_{31} - 2\sigma_2)^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 \right], \\ & + x_3 \left[3\sigma_{12}^2 + (\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 \right] + x_4 \left[3\sigma_{31}^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 \right], \\ & + x_5 \left[\frac{1}{3} \left(\frac{3\sqrt{3}}{\sqrt{5}} \sigma_{23} - 2\sigma_3 + \sigma_{12} \right)^2 + \left(\frac{\sqrt{3}}{\sqrt{5}} (\sigma_{2\hat{3}} + \sigma_{3\hat{2}}) + \sigma_{1\hat{2}} - \sigma_{2\hat{1}} \right)^2 \right] + \frac{4}{3}x_6 (2\sigma_{1\hat{1}} + 3\sigma_{2\hat{2}})^2, \end{aligned} \quad (6.23)$$

provided that the constants are chosen (up to scale) so that

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, \frac{1}{3}, \frac{1}{3}, \frac{5}{14}, \frac{3}{28}). \quad (6.24)$$

There is no freedom, except for an overall scaling, in the choice of the metric coefficients, the embedding is isotropy irreducible, and thus it is Einstein [9]. We find the Einstein constant and the invariant I_2 are given by

$$\lambda = \frac{43}{28}, \quad I_2 = \frac{69883}{3698}. \quad (6.25)$$

6.6 $G_2/U(2)$ flag manifold

There are two $G_2/U(2)$ cosets that one may consider, in which the $U(2)$ is taken to be either $SU(2)_L \times U(1)_R$, or else $SU(2)_R \times U(1)_L$. (The $U(1)$ factors are taken from $SU(2)_R$ or $SU(2)_L$ respectively.) The case when $U(2)$ is $SU(2)_L \times U(1)_R$ gives rise to the flag manifold $G_2/U(2)$.

The metric on the coset $G_2/[SU(2)_L \times U(1)_R]$ is obtained by dividing out the y_4 terms and the last of the three y_3 terms in (5.9):

$$\begin{aligned} ds_{10}^2 = & y_1 [(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} - \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2] \\ & + y_2 [(\sigma_{1\hat{2}} + \sigma_{2\hat{1}})^2 + (\sigma_{1\hat{3}} + \sigma_{3\hat{1}})^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2] \\ & + y_3 [(\sigma_{23} - \sigma_1)^2 + (\sigma_{31} - \sigma_2)^2] \\ & + y_5 (\sigma_{1\hat{1}} - \sigma_{2\hat{2}})^2 + y_6 (\sigma_{1\hat{1}} + \sigma_{2\hat{2}})^2. \end{aligned} \quad (6.26)$$

This factoring can be performed provided that $y_1 = y_6 = 3y_2 = 3y_5$. The Einstein equations then imply that $y_3 = 2y_1$ or $y_3 = \frac{2}{3}y_1$. Thus, up to scaling, we obtain the two inequivalent

Einstein metrics

$$(y_1, y_2, y_3, y_5, y_6) = (3, 1, 6, 1, 3) : \quad \lambda = \frac{1}{2}, \quad I_1 = \frac{27}{16}, \quad I_2 = \frac{460}{27}, \\ (y_1, y_2, y_3, y_5, y_6) = (3, 1, 2, 1, 3) : \quad \lambda = \frac{11}{18}, \quad I_1 = \frac{161051}{104976}, \quad I_2 = \frac{2020}{121}. \quad (6.27)$$

The coset $G_2/U(2)$ that we have constructed here, denoted by $G_2/U(2)_3$ in [3], is the flag manifold of G_2 . (The subscript is the Dynkin index of the $SU(2)$ factor in the denominator subgroup.) The two Einstein metrics were obtained first in [12], and were recently discussed further in [13].

6.7 $G_2/U(2)$ Grassmann manifold $G_2^+(\mathbb{R}^7)$

The other $G_2/U(2)$ coset is obtained by taking $U(2) = SU(2)_T \times U(1)_L$, and is thus denoted by $G_2/U(2)_1$ in [3]. As discussed in [14], this $U(2)$ subgroup of G_2 is also contained in the $SU(3)$ subgroup of G_2 (in fact it is the intersection of the $SU(3)$ and the $SU(2) \times SU(2)$ subgroups of G_2). The resulting coset space is isomorphic to the Grassmannian $G_2^+(\mathbb{R}^7) = SO(7)/[SO(2) \times SO(5)]$ of oriented 2-planes through the origin in \mathbb{R}^7 [14].

The $SU(3)$ subgroup of G_2 is spanned by the left-invariant 1-forms $E_4^\pm, E_5^\pm, E_6^\pm, H_1$ and H_2 (see (5.4) and (5.5)). The $U(2)$ subgroup can be taken to be spanned by E_4^\pm and H_1 (spanning $SU(2)$) together with H_2 (spanning the $U(1)$ factor). Thus we may write G_2 -invariant metrics on the Grassmannian $G_2/U(2) = G_2^+(\mathbb{R}^7)$ by dividing out the terms proportional to x_6, x_7 and x_8 in (5.7). This truncation is consistent provided that we take $x_1 = x_2$ and $x_4 = x_5$, and so we consider metrics of the form

$$ds_{10}^2 = y_1 \left[(\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2 + \frac{1}{3}(\sigma_{23} - 2\sigma_1)^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 + \frac{1}{3}(\sigma_{31} - 2\sigma_2)^2 \right] \\ + y_2 \left[(\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 + \frac{1}{3}(\sigma_{12} - 2\sigma_3)^2 \right] \\ + y_3 \left[3\sigma_{23}^2 + (\sigma_{2\hat{3}} + \sigma_{3\hat{2}})^2 + 3\sigma_{31}^2 + (\sigma_{3\hat{1}} + \sigma_{1\hat{3}})^2 \right]. \quad (6.28)$$

Scaling so that $y_1 = 1$, we find that the Einstein equations imply

$$y_3 = \frac{(3y_2 - 2)(y_2 + 2)}{2(5y_2^2 - 18y_2 + 8)}, \quad (6.29)$$

and y_2 must satisfy

$$(y_2 - 2)(60y_2^5 - 776y_2^4 + 1891y_2^3 - 1570y_2^2 + 523y_2 - 56) = 0. \quad (6.30)$$

The quintic has three real roots, all of which are positive:

$$y_2 \approx 0.1868941, \quad y_2 \approx 1.67467, \quad y_2 \approx 10.047. \quad (6.31)$$

The first two of these, and the root $y_2 = 2$ in (6.30), all give positive values for y_3 and thus yield Riemannian Einstein metrics. These have λ , I_1 and I_2 given by

$$\begin{aligned} (y_1, y_2, y_3) &= (1, 2, 1) : \quad \lambda = \frac{5}{6}, \quad I_1 = \frac{3125}{3888}, \quad I_2 = \frac{26}{3}, \\ (y_1, y_2, y_3) &\approx (1, 0.1868941, 0.327159) : \quad \lambda \approx 1.94012, \\ &\qquad\qquad\qquad I_2 \approx 0.549861, \quad I_2 \approx 30.4872, \\ (y_1, y_2, y_3) &\approx (1, 1.67467, 0.684128) : \quad \lambda \approx 1.00414, \\ &\qquad\qquad\qquad I_1 \approx 0.80014, \quad I_2 \approx 10.8238. \end{aligned} \quad (6.32)$$

(We take the volume to be $V = y_1^2 y_2 y_3^2$.) These metrics were found in [15, 16], and discussed further in [14]. The first metric is just the standard $SO(7)$ -invariant metric on the Grassmannian $SO(7)/[SO(2) \times SO(5)]$ [14].

The third root in (6.31), for which y_3 is negative, gives a pseudo-Riemannian Einstein metric of signature $(4, 6)$:

$$\begin{aligned} (y_1, y_2, y_3) &\approx (1, 10.046978, -0.510773) : \quad \lambda \approx 0.312078, \\ &\qquad\qquad\qquad I_1 \approx 0.007759, \quad I_2 \approx -140.7999. \end{aligned} \quad (6.33)$$

6.8 $G_2/SU(3) = S^6$

There is an $SU(3)$ maximal subgroup of G_2 , for which the associated left-invariant 1-forms are

$$\sigma_{23}, \quad \sigma_{31}, \quad \sigma_{12}, \quad (\sigma_{2\hat{3}} + \sigma_{3\hat{2}}), \quad (\sigma_{3\hat{1}} + \sigma_{1\hat{3}}), \quad (\sigma_{1\hat{2}} + \sigma_{2\hat{1}}), \quad (\sigma_{1\hat{1}} - \sigma_{2\hat{2}}), \quad (\sigma_{1\hat{1}} + \sigma_{2\hat{2}}). \quad (6.34)$$

We find that there is a unique (up to overall scale) Einstein metric on $G_2/SU(3)$, given by

$$\begin{aligned} ds_6^2 &= (\sigma_{2\hat{3}} - \sigma_{3\hat{2}})^2 + (\sigma_{3\hat{1}} - \sigma_{1\hat{3}})^2 + (\sigma_{1\hat{2}} - \sigma_{2\hat{1}})^2 \\ &\quad + \frac{1}{3}(\sigma_{23} - 2\sigma_1)^2 + \frac{1}{3}(\sigma_{31} - 2\sigma_2)^2 + \frac{1}{3}(\sigma_{12} - 2\sigma_3)^2. \end{aligned} \quad (6.35)$$

This is S^6 , with its standard Einstein metric. (With the scaling we have chosen, it has $\lambda = 5/3$.)

7 Conclusion

In this paper, we have found four inequivalent positive-definite Einstein metrics on the group manifold $SO(5)$, and six inequivalent positive-definite Einstein metrics on G_2 . Two

of the metrics on $SO(5)$, and four of the metrics on G_2 , appear to be new. One motivation for studying this question was the possible utility of such metrics for the construction of background solutions in supergravity, string theory and M-theory.

Mindful of the possible applications to the Chronology Protection Conjecture, and related issues concerning closed timelike curves (CTCs), we also searched for Einstein metrics of indefinite signature. We found one on $SU(3)$ with signature $(2, 6)$, one on $G_2/SU(2)^{\text{diag}}$ with signature $(6, 5)$, and one on the Grassmannian $G_2^+(\mathbb{R}^7) = G_2/U(2)$ with signature $(4, 6)$. The absence of Lorentzian examples is striking, since there is no topological obstruction to a group manifold, compact or otherwise, admitting a Lorentzian metric (although it will, if compact, have CTCs). Indeed, the necessary and sufficient condition that a manifold admit a time orientable Lorentzian metric is that it admit an everywhere non-vanishing vector field, or, equivalently, that the Euler number vanish. This condition holds trivially for group manifolds, and for all odd-dimensional compact manifolds. Of course, if we relaxed the Einstein condition it would be trivial to write down Lorentzian metrics on group manifolds, simply by taking the matrix x_{ab} in (1.2) to have one negative eigenvalue.

It is possible that the absence of Lorentzian metrics may be ascribed to the restricted nature of our ansätze. It may also be, by analogy with the Gödel solution, that to obtain Lorentzian metrics satisfying the Einstein equations, one needs to add material sources such as a perfect fluid. This is an interesting topic for future investigation.

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